Finding a Sparse Vector in a Subspace: Linear Sparsity Using Alternating Directions

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Problem Statement:
- Given a sparse vector $x_0$ embedded in an $n$-dimensional subspace $S \subseteq \mathbb{R}^p$, provided any basis of $S$, can we efficiently recover $x_0$?
- Equivalently, provided a matrix $A \in \mathbb{R}^{(p-n) \times p}$ whose row span forms the null space of $S$, can we solve
  \begin{equation}
  \min \{ |x|_0 \mid A^T x = 0 \} \quad \text{s.t.} \quad Ax = b
  \end{equation}

Motivation:
- In contrast to the standard sparse recovery problem $\min \{ |x|_0 \mid Ax = b \}$, convex relaxation works nearly optimally for generic design of $A$, the computational property of (1) is nearly as well understood.
- Variants of (1) have been studied in numerical linear algebra, sparse PCA, blind source separation, dictionary learning (DL), spectral estimation and Pons’ Problem, and graphical model learning.

Existing Work:
- $\ell^1/\ell^q$ Recovery [Spielman et al.] and [Hand et al.]:
- Semi-Definite Programming (SDP) Relaxation: $\min \{ |x|_0 \mid \langle A^T A, x \rangle = 0, \|x\|_q \leq \epsilon \}$.
- Sum-of-Squares (SOS) Relaxation [Barak et al.]:
  \begin{align*}
  &\text{Method} \quad \text{Recovery Condition} \quad \text{complexity} \quad \text{SOS} \quad \text{p.d.} \quad \text{poly}(p) \quad \\
  &\ell^1/\ell^1 \quad \theta \geq C(1/n) \quad \Theta(\sqrt{n}p) \quad \Theta(n) \quad \text{high order poly}(p)
  \end{align*}

Question 1: Is there a practical algorithm that provably recovers a sparse vector with $\theta \geq 1/\sqrt{q}f$ from a generic subspace $S$?

Contributions of this Work:
- Proposed a simple ADM algorithm, addressed the problem under the PSV model, exact recovery for $x_0$, having $p \gg \Theta(n \log n)$. Provided $p \geq \Theta(n^2 \log n)$.
- Performs well empirically – succeeds for both the PSV and DL models, with $p \geq \Theta(n \log n)$.

Problem Formulation and Optimality Conditions

- Planted Sparse Vector (PSV) Model: A single sparse vector $x_0$ embedded in an otherwise random subspace:
  \begin{align}
  S &= \text{span} \{ x_0, g_1, \ldots, g_m \} \\
  \text{where} \quad x_0 &= \text{sign}(x_1) \quad \text{and} \quad x_1 = \frac{|x_1|}{\|x_1\|_2} \text{Ber}(\theta)
  \end{align}

- Nonconvex $\ell^q/\ell^q$ Minimization Problem:
  \begin{align}
  \min \{ |x|_1 \mid \text{s.t.} \quad x \in S, \|x\|_q = 1 \}
  \end{align}
  which is equivalent to
  \begin{align}
  \min \{ |y|_1 \mid \text{s.t.} \quad \|y\|_q = 1 \}
  \end{align}

- Where $Y \in \mathbb{R}^{p \times p}$ is an arbitrary orthonormal matrix whose columns form a basis of $S$.

- Theorem (Global Optimality for $\ell^q/\ell^q$ Recovery): Suppose $\hat{s}$ follows the PSV model, and $q_\theta$ be the optimum (2), with very high probability, we have $\|q_\theta - q_\theta^0\|_F \leq k_p$ for some $\theta \not\in \Theta(0)$, provided
  \begin{align}
  p \geq (1/(\theta \log n)) \quad \text{and} \quad \theta \leq \sqrt{q}
  \end{align}

Algorithm based on Alternating Direction Method (ADM)

- Alternating Minimization: Consider a relaxation of (2):
  \begin{align}
  \min_{x, y} \{ \|y - x\|_1 + \lambda \|x_1\|_q \mid y \in S \}
  \end{align}

- Closed form solutions of (3), (4) lead to one ADM iteration
  \begin{align}
  \begin{bmatrix}
  y^{(k+1)} \\
  Y^{(k+1)} \end{bmatrix} &= \begin{bmatrix}
  S^T \\
  S \end{bmatrix} \begin{bmatrix}
  q^{(k+1)} \\
  Q^{(k+1)} \end{bmatrix}
  \end{align}

Discussions

- More Application Ideas?
- Intriguing Experiments on Dictionary Learning:

- Efficient algorithms can also achieve linear sparsity regime for the squared dictionary learning under the semi-Supervised Gaussian model.
- Generalization: Can we develop general tools for
  \begin{align}
  \min_{w \in M} \frac{1}{p} \sum_{i=1}^{p} f_i(w_i) \quad \text{s.t.} \quad w \in M
  \end{align}
  $f_i(w_i)$: nonconvex function, $M$: smooth manifold.
- NonConvex Problems as a Whole: Phase retrieval, matrix/tensor completion, robust PCA, blind deconvolution, etc.